Connection probabilities for Ising model and their relation to Dyson's circular ensemble

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Outline

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Ising model

Ising model [Lenz 1920]

A model for ferromagnet, to understand the phase transition.

- $G = (V, E)$ a finite graph
- $\sigma \in \{\ominus, \oplus\}^V$
- $H(\sigma) = -\sum_{x\sim y} \sigma_x \sigma_y$

Ising model is the probability measure of inverse temperature $\beta > 0$:

$$\mu_{\beta,G}[\sigma] \propto \exp(-\beta H(\sigma))$$

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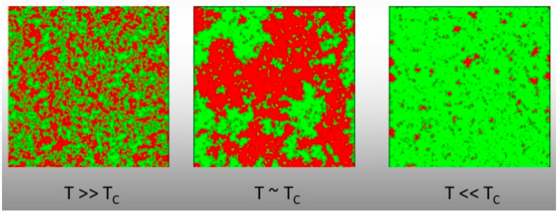
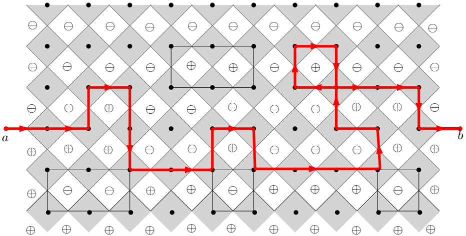


- $\beta > \beta_c$: ordered
- $\beta \approx \beta_c$: critical
- $\beta < \beta_c$: chaotic

Conformal invariance of interfaces



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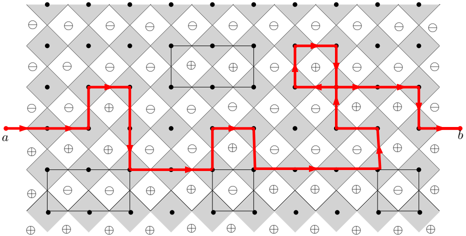

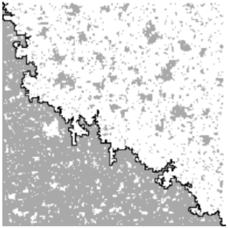

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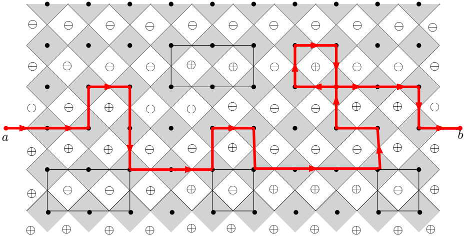
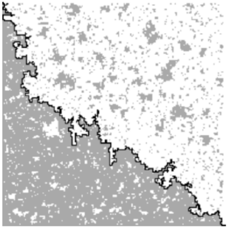
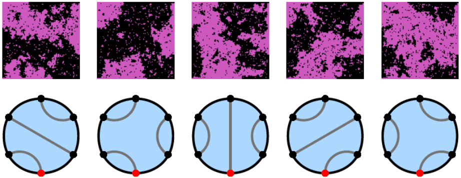




Theorem [Chelkak-Smirnov et al. Invent. 2012]

The interface in critical Ising model on $\mathbb{Z}^2$ with Dobrushin boundary conditions converges weakly to SLE_3.

Connection probabilities for Ising model



Theorem [Peltola-W. AAP 2023+]

The connection of Ising interfaces forms a planar link pattern $\mathcal{A}_\delta$.

$$\lim_{\delta\to 0}\mathbb{P}[\mathcal{A}_\delta=\alpha]=\frac{\mathcal{Z}_\alpha(\Omega;x_1,\ldots,x_{2N})}{\mathcal{Z}_{Ising}(\Omega;x_1,\ldots,x_{2N})},\quad \mathcal{Z}_{Ising}=\sum_{\alpha\in\mathrm{LP}_N}\mathcal{Z}_\alpha,$$

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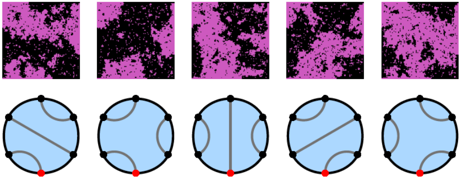

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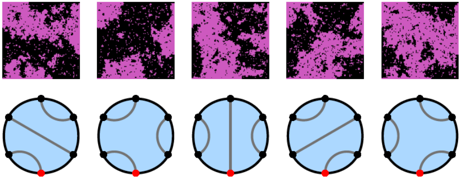

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- Related to correlation functions in CFT.

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PDE : $\left[\frac{\kappa}{2}\partial_i^2 + \sum_{j\neq i}\left(\frac{2}{x_j - x_i}\partial_j - \frac{2h}{(x_j - x_i)^2}\right)\right]\mathcal{Z}(x_1, \ldots, x_{2N}) = 0$, where $h = (6-\kappa)/2\kappa$.

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- PDE : 2N variables, 2N PDEs
- ASY : boundary value ?

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Questions

Existence and uniqueness ?

Pure partition functions

Uniqueness [Flores-Kleban, CMP 2015]

Fix $\kappa \in (0, 8)$. If there exist collections of smooth functions satisfying PDE, COV and ASY, they are (essentially) unique.

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Existence

- $\kappa \in (0,8) \setminus \mathbb{Q}$ [Kytölä-Peltola, CMP 2016]
- $\kappa \in (0,4]$ [Peltola-W. CMP 2019, Beffara-Peltola-W. AOP 2021]
- $\kappa \in (0,6]$ [W. CMP 2020]

- Coulumb gas techniques
- Global multiple SLEs
- Hypergeometric SLE

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Theorem [W. CMP 2020]

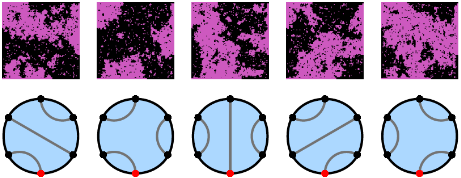
Fix $\kappa \in (0, 6]$. The pure partition functions are the recursive collection $\{\mathcal{Z}_\alpha : \alpha \in \cup_N \mathrm{LP}_N\}$ of smooth functions $\mathcal{Z}_\alpha : \mathfrak{X}_{2N} \to \mathbb{R}$ uniquely determined by the following properties :

PDE, COV, ASY as well as **PLB** :

$$0 < \mathcal{Z}_\alpha(x_1, \ldots, x_{2N}) \le \prod_{\{a,b\}\in\alpha} |x_b - x_a|^{-2h}, \quad \forall (x_1, \ldots, x_{2N}) \in \mathfrak{X}_{2N}.$$

$\{\mathcal{Z}_\alpha : \alpha \in \mathrm{LP}_N\}$ is linearly independent and forms a basis for the solution space.

Connection probabilities for Ising model



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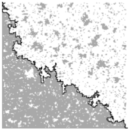
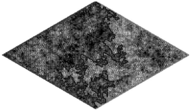
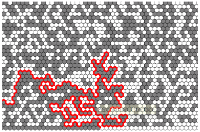
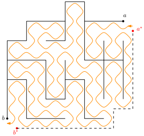
Conformal invariance in 2D critical lattice models



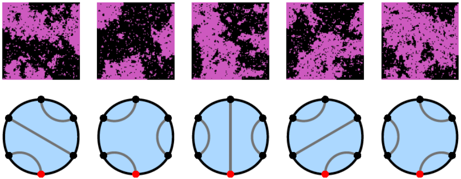






- Loop-erased random walk (LERW) : $\kappa = 2$ [Lawler-Schramm-Werner, AOP 2004]
- Ising model : $\kappa = 3$ [Chelkak-Smirnov et al. 2012]
- Level lines of GFF : $\kappa = 4$ [Schramm-Sheffield, ACTA 2009]
- FK-Ising model : $\kappa = 16/3$ [Chelkak-Smirnov et al. 2012]
- Percolation : $\kappa = 6$ [Smirnov 2001]
- Uniform spanning tree (UST) : $\kappa = 8$ [Lawler-Schramm-Werner, AOP 2004]

Connection probabilities



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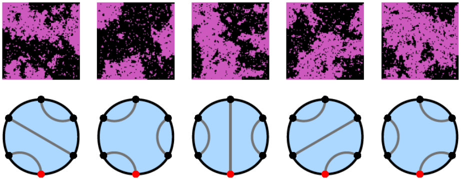

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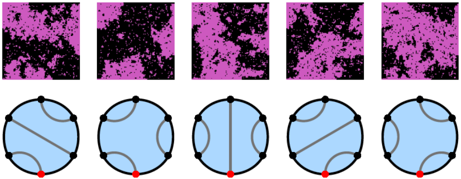

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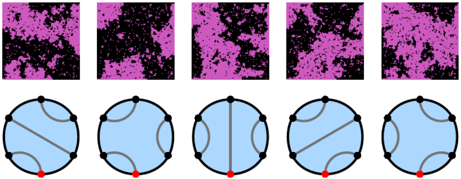

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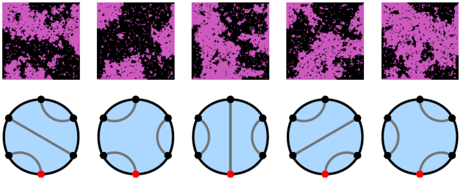

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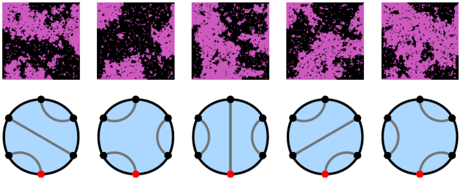

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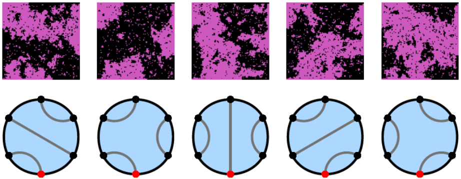

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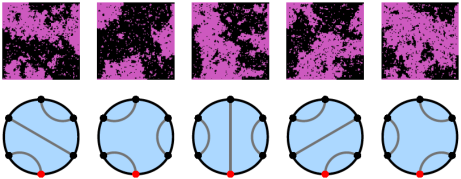

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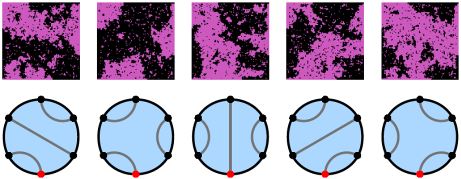

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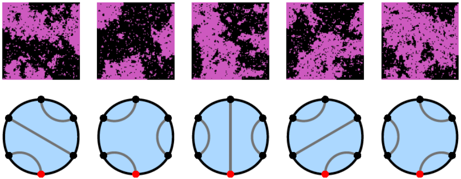

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Consequence : given the connectivity α, a single interface $\sim$ Loewner chain associated to $\mathcal{Z}_\alpha$:

$$\mathrm{d}W_t = \mathrm{d}B_t + \partial_j \log \mathcal{Z}_\alpha(V_t^1, \ldots, V_t^{j-1}, W_t, V_t^{j+1}, \ldots, V_t^{2N})\mathrm{d}t.$$

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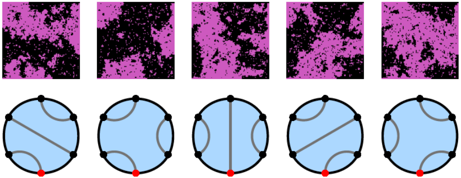

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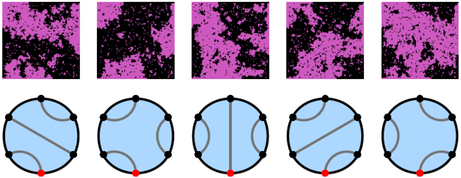

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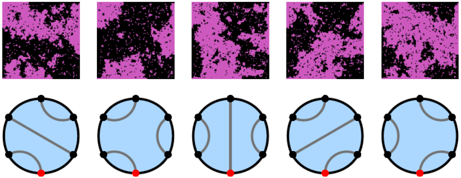

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Proposition [Feng-W.-Yang 2023]

Fix $\kappa \in (0, 4]$ and $a = 2/\kappa$. The solution $\boldsymbol{\theta}_t = (\theta_t^1, \ldots, \theta_t^{2N})$ to (1) conditioned on $\{T > s\}$ converges in total variation distance as $s \to \infty$ to $2N$ radial Bessel process

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whose invariant density is called Dyson's circular ensemble [Dyson, J. Math. Phys. 1962] :

$$f(\theta^1, \ldots, \theta^{2N}) \propto \prod_{1 \leq j < k \leq 2N} |\sin(\theta^k - \theta^j)|^{4a}. \tag{3}$$

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Key ingredients : [Peltola-W. CMP 2019], [Healey-Lawler, PTRF 2021], $2N$-time local martingale :

$$M_{\boldsymbol{t}}^{\alpha} = g_{\boldsymbol{t}}'(0)^{-2N\tilde{b}} \prod_{j=1}^{2N} h_{\boldsymbol{t},j}'(\xi_{t_j}^j)^b g_{\boldsymbol{t},j}'(0)^{\tilde{b}} \times \mathcal{G}_\alpha(\theta_{\boldsymbol{t}}^1, \ldots, \theta_{\boldsymbol{t}}^{2N}) \exp\left(\frac{c}{2} \sum_{j=1}^{2N} \mu_{\boldsymbol{t}}^j\right).$$

Applications : estimates for multiple SLEs

Theorem [Feng-W.-Yang 2023]

Fix $\kappa \in (0,4]$. Fix $\theta^1 < \cdots < \theta^{2N} < \theta^1 + \pi$ and write $\boldsymbol{\theta} = (\theta^1, \ldots, \theta^{2N})$. We denote by $(\gamma_1, \ldots, \gamma_N) \sim \mathbb{P}_\alpha^{(\boldsymbol{\theta})}$ the law of chordal N-SLE$_\kappa$ in polygon $(\mathbb{D}; \exp(2\mathrm{i}\theta^1), \ldots, \exp(2\mathrm{i}\theta^{2N}))$ associated to link pattern $\alpha \in \mathrm{LP}_N$. We have

$$\mathbb{P}_\alpha^{(\boldsymbol{\theta})}\left[\mathrm{dist}(0,\gamma_j) < r, 1 \le j \le N\right] = CG_\alpha(\boldsymbol{\theta}) r^{A_{2N}}(1 + O(r^u)), \quad \text{as } r \to 0+,$$

where dist is Euclidean distance.

- A_{2N} is $2N$-arm exponent :
$$A_{2N} = \frac{16N^2 - (4-\kappa)^2}{8\kappa};$$
- G_α is Green's function for chordal N-SLE$_\kappa$:
$$G_\alpha(\theta^1, \ldots, \theta^{2N}) := \frac{\mathcal{G}_*(\theta^1, \ldots, \theta^{2N})}{\mathcal{G}_\alpha(\theta^1, \ldots, \theta^{2N})},$$
  where $\mathcal{G}_*$ is the partition function for $2N$-sided radial SLE$_\kappa$:
$$\mathcal{G}_*(\theta^1, \ldots, \theta^{2N}) := \prod_{1 \le j < k \le 2N} |\sin(\theta^k - \theta^j)|^{2/\kappa};$$
- $C \in (0,\infty)$ is a constant depending on κ, N, α and $u > 0$ is a constant depending on κ, N.

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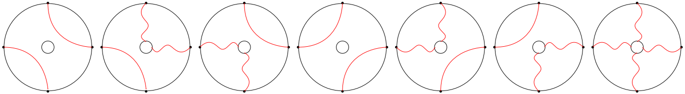
- Proved for $N = 1$ and $\kappa \in (0,8)$: [Lawler-Rezaei, AOP 2015].
- Proved for $N = 2$ and $\kappa \in (0,8)$: [Zhan, CMP 2020].
- We prove it for all $N \ge 1$ and α and $\kappa \in (0,4]$.

Key ingredients : [Peltola-W. CMP 2019], [Healey-Lawler, PTRF 2021].

Applications : multiple Ising interfaces in annulus

We consider critical Ising model in annulus with boundary conditions :

alternating $\oplus/\ominus$ on the outer boundary and free on the inner boundary.



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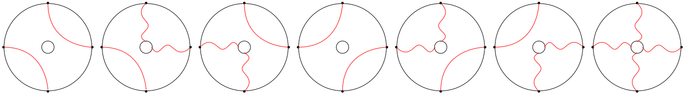

Theorem [Feng-W.-Yang 2023]

Suppose $(\eta_1, \ldots, \eta_{2N}) \sim \mathbb{P}^{(\boldsymbol{\theta})}_{\text{Ising}}$. We have

$$\mathbb{P}^{(\boldsymbol{\theta})}_{\text{Ising}}[\eta_1, \ldots, \eta_{2N} \text{ all hit } r\mathbb{D}] = r^{\frac{16N^2-1}{24}+o(1)}, \quad \text{as } r \to 0. \tag{4}$$

Conditioned on the event $\{\eta_1, \ldots, \eta_{2N} \text{ all hit } r\mathbb{D}\}$, the law $\mathbb{P}^{(\boldsymbol{\theta})}_{\text{Ising}}$ converges in total variation distance to $2N$-sided radial SLE_3 whose driving function is $2N$ radial Bessel process

$$\mathrm{d}\theta^j_t = \mathrm{d}B^j_t + \frac{4}{3}\sum_{k\neq j}\cot(\theta^j_t - \theta^k_t)\mathrm{d}t, \quad 1 \leq j \leq 2N. \tag{5}$$

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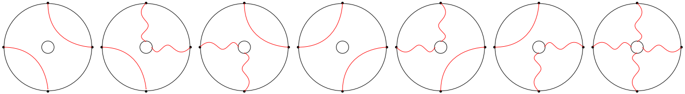

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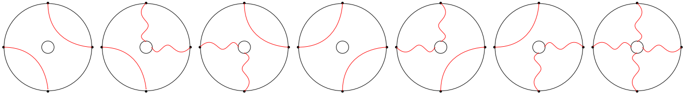

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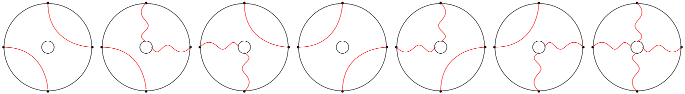

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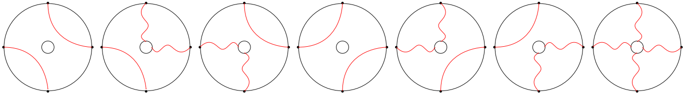

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Thanks !

1. [Peltola-W. CMP 2019] Global and local multiple SLEs for $\kappa \leq 4$ and connection probabilities for level lines of GFF. *Comm. Math. Phys.* 366(2) : 469-536, 2019.
2. [W. CMP 2020] Hypergeometric SLE : conformal Markov characterization and applications *Comm. Math. Phys.* 374(2) : 433-484, 2020.
3. [Beffara-Peltola-W. AOP 2021] On the uniqueness of global multiple SLEs *Ann. Probab.* 49(1) : 400-434, 2021.
4. [Liu-W. EJP 2021] Scaling limits of crossing probabilities in metric graph GFF *Electron. J. Probab.* 26 : article no. 37, 1-46, 2021.
5. [Ding-Wirth-W. AIHP 2022] Crossing estimates from metric graph and discrete GFF *Ann. Inst. H. Poincaré Probab. Statist.* 58(3) :1740-1774, 2022.
6. [Liu-W. Bernoulli 2023] Loop-erased random walk branch of uniform spanning tree in topological polygons. *Bernoulli.* 29(2) : 1555-1577, 2023.
7. [Peltola-W. AAP 2023+] Crossing probabilities of multiple Ising interfaces *Ann. Appl. Probab.* to appear. 2023+
8. [Han-Liu-W. 2020] Hypergeometric SLE with $\kappa = 8$: convergence of UST and LERW in topological rectangles. arXiv :2008.00403. 2020.
9. [Liu-Peltola-W. 2021] Uniform spanning tree in topological polygons, partition functions for SLE(8), and correlations in $c = -2$ logarithm CFT. arXiv :2108.04421. 2021.
10. [Feng-Peltola-W. 2022] Connection probabilities of multiple FK-Ising interfaces. arXiv :2205.08800. 2022.
11. [Feng-W.-Yang 2023] Multiple Ising interfaces in annulus and $2N$-sided radial SLE. arXiv :2302.09124. 2023.